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**LETTER TO THE EDITOR**

# Decorating random quadrangulations

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**Abstract**
On various regular lattices (simple cubic, body centred cubic, etc) decorating an edge with an Ising spin coupled by bonds of strength $L$ to the original vertex spins and competing with a direct anti-ferromagnetic bond of strength $\alpha L$ can give rise to three transition temperatures for suitable $\alpha$. The system passes through ferromagnetic, paramagnetic, anti-ferromagnetic and paramagnetic phases respectively as the temperature is increased. For the square lattice on the other hand, multiple decoration is required to see this effect. We note here that a single decoration suffices for the Ising model on planar random quadrangulations (coupled to 2D quantum gravity). Other random bipartite lattices such as the Penrose tiling are more akin to the regular square lattice and require multiple decoration to have any affect.

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The application of various classes of transformations, such as duality, star–triangle, and decoration–iteration to the Ising model on regular two-dimensional lattices has been exhaustively investigated in the past [1] both as a means of obtaining solutions on new lattices and as a way of modelling the physical properties of substances with more complicated behaviour than simple ferromagnetism. The decoration–iteration transformation acts exclusively at the level of the bonds in the lattice, so it knows nothing about the larger-scale structure. It is therefore still applicable on lattices which exhibit some form of geometrical disorder. In this short note we look at the effect of a decoration–iteration transformation on an Ising model on just such a class of lattices. We consider a model with competing ferromagnetic and anti-ferromagnetic interactions living on an ensemble of random quadrangulations (i.e. coupled to two-dimensional quantum gravity).

The basic decoration–iteration transform is shown in figure 1, where summing over the central spin $s$ with couplings $L$ gives rise to a new effective coupling $K$ between the primary vertex spins $\sigma_1, \sigma_2$:

$$\sum_s \exp\left[Ls(\sigma_1 + \sigma_2)\right] = A \exp(K\sigma_1\sigma_2) \tag{1}$$

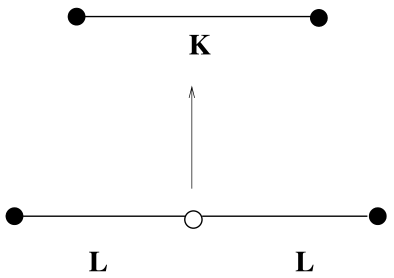


**Figure 1.** Summing over the central spin values gives an effective coupling $K$.

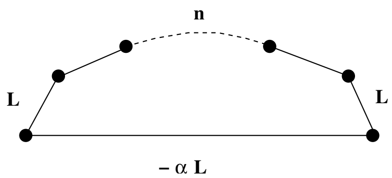


**Figure 2.** A direct antiferromagnetic bond of strength $-\alpha L$ decorated by $n$ spins coupled by ferromagnetic bonds of strength $L$.

where $A = 2(\cosh(2L))^{1/2}$ and $\exp(2K) = \cosh(2L)$, i.e.

$$K = \tfrac{1}{2}\log\left[\cosh(2L)\right]. \tag{2}$$

If a direct anti-ferromagnetic bond is stirred into the mix as well for good measure this becomes

$$K = -\alpha L + \tfrac{1}{2}\log\left[\cosh(2L)\right]. \tag{3}$$

This may easily be extended to a situation such as that shown in figure 2 where we introduce $n$ decorating spins as well as the direct anti-ferromagnetic bond. In this case summing over the intermediate spins gives

$$K = -\alpha L + \frac{1}{2}\log\left[\frac{(\exp(2L)+1)^{n+1} + (\exp(2L)-1)^{n+1}}{(\exp(2L)+1)^{n+1} - (\exp(2L)-1)^{n+1}}\right] \tag{4}$$

along with an equation for the normalization factor

$$A^2 = 2^n \frac{(\sinh(2L))^{n+1}}{\sinh(2K)}. \tag{5}$$

It was pointed out by Nakano [2] that generically the form of the transformation in both equations (3) and (4) meant that *three* transitions could occur for an Ising model. Whether this behaviour actually occurred or not depended on the critical temperature values for a given lattice, along with the value of $\alpha$ and the degree $n$ of iteration. For the singly decorated model, the minimum value of $K$ in equation (3), for instance, is $K_{\min} = -\frac{1}{2}\log(2) \sim -0.3196\ldots$ which is attained as $\alpha \to 1$. Since this is larger than the critical value of the coupling at the anti-ferromagnetic transition on the square lattice $K_{\mathrm{crit}} = -0.446\,09\ldots$ the $K(L)$ curve can at best intersect the ferromagnetic transition value $K_{\mathrm{crit}} = +0.446\,09\ldots$ and only one (ferromagnetic) transition will be in evidence in the decorated model. A generic curve displaying this behaviour is shown in figure 3, which is for $\alpha = \frac{4}{5}$. We can see that $K(L)$ does not dip below the line at $K_{\mathrm{crit}} = -0.446\,09\ldots$, but does cross $K_{\mathrm{crit}} = +0.446\,09\ldots$. As $\alpha$ is increased still further, $K(L)$ eventually becomes monotone decreasing, attaining $K_{\min}$ as $\alpha \to 1$, so even the ferromagnetic transition disappears.

This behaviour is strongly lattice dependent since we are looking for intersections of the $K(L)$ curve with the critical values of the coupling. Since $K_{\mathrm{crit}} = \pm 0.2217\ldots$ for the simple cubic lattice and $K_{\mathrm{crit}} = \pm 0.1574$ for the body-centred cubic lattice, these will display multiple transitions when the $K(L)$ curve cuts through the anti-ferromagnetic critical coupling values for sufficiently large $\alpha$. As $L$ is decreased (i.e. the temperature is increased), following the $K(L)$ curve then takes one from a ferromagnetic phase via a paramagnetic phase to an anti-ferromagnetic phase and finally to a paramagnetic phase again. One thus has a sequence of one ferromagnetic and two anti-ferromagnetic transitions.

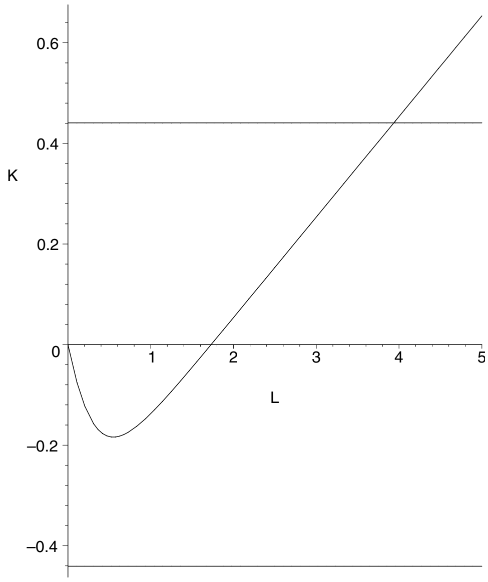


**Figure 3.** For the square lattice ising model a singly-decorated bond as in figure 1, will *not* give multiple transition points since the curve $K(L)$ only crosses the ferromagnetic value $K = 0.446\,09\ldots$. We have taken $\alpha = \frac{4}{5}$ in the figure.

In short, for a lattice with an anti-ferromagnetic critical coupling of sufficiently small modulus, $|K_{\mathrm{crit}}| < |K_{\mathrm{min}}|$, multiple transitions are to be expected when competing direct and decorated bonds are present. For $n$-fold decoration the minimum value of $K(L)$ tends to $K_{\mathrm{min},\,n} = -\frac{1}{2}\log(n+1)$ as $\alpha \to 1$, so $n = 2$ is sufficient to induce a triple transition in even the square lattice Ising model.

We now turn to the case of the Ising model coupled to 2D quantum gravity as an example of the application of decoration–iteration transformations on geometrically disordered lattices. The partition function for the Ising model on a single planar graph $G^n$ with $n$ vertices is just [3]

$$Z_{\mathrm{single}}(G^n, K) = \sum_{\{\sigma\}} \exp\left( K \sum_{<i,j>} \sigma_i \sigma_j \right). \tag{6}$$

The coupling to gravity is incorporated by introducing a sum over some class of planar graphs $\{G^n\}$

$$Z_n(K) = \sum_{\{G^n\}} Z_{\mathrm{single}}(G^n, K). \tag{7}$$

The grand canonical partition function for this model

$$\mathcal{Z} = \sum_{n=1}^{\infty} \left( \frac{-4gc}{(1-c^2)^2} \right)^n Z_n(K) \tag{8}$$

where $c = \mathrm{e}^{-2K}$ can be expressed as the free energy

$$\mathcal{Z} = -\frac{1}{N^2} \log \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \, \exp\left( -\mathrm{Tr}\left[ \frac{1}{2}(\phi_1^2 + \phi_2^2) - c\phi_1\phi_2 - \frac{g}{4}(\phi_1^4 + \phi_2^4) \right] \right) \tag{9}$$

of a matrix model, where we have specifically given the potential which generates $\phi^4$ graphs and $\phi_{1,2}$ are $N \times N$ Hermitian matrices. The $N \to \infty$ limit is taken to pick out planar graphs.

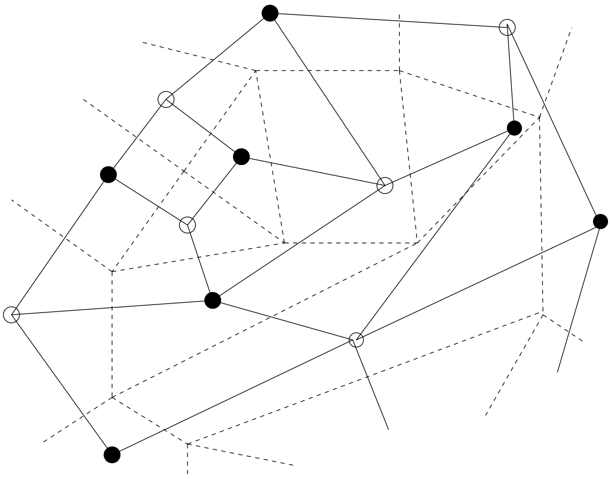

**Figure 4.** A section of a planar $\phi^4$ graph and its dual quadrangulation. As the vertex shading makes clear, the quadrangulation, although random, is still bipartite and hence will admit an antiferromagnetic transition.

The model as defined above has spins living on the vertices of $\phi^4$ planar graphs and has a critical coupling of $c_{\mathrm{crit}} = \frac{1}{4}$, $K_{\mathrm{crit}} = 0.693\,14\ldots$ [3]. It displays no anti-ferromagnetic transition, because both odd and even loops are present. The decoration process outlined above will thus not induce multiple transitions since there is no anti-ferromagnetic line for the curve $K(L)$ to cross. It is a similar story for both $\phi^3$ planar graphs and their dual triangulations.

However, the dual to the $\phi^4$ graphs, random quadrangulations, are rather more amenable to decoration. We can see from figure 4 that, although the number of squares round a vertex is arbitrary, the fact that every face is a square means that the random quadrangulation is bipartite and hence would be expected to have an anti-ferromagnetic transition. This has been confirmed both by direct simulation [4] and by studies of the Fisher (temperature) zeroes of the partition function [5, 6]. The critical coupling for the ferromagnetic transition on the random quadrangulations is given by the dual of the $\phi^4$ value, namely $c^*_{\mathrm{crit}} = (1 - c_{\mathrm{crit}})/(1 + c_{\mathrm{crit}}) = \frac{3}{5}$ and the anti-ferromagnetic value by its inverse, $\frac{5}{3}$. If we translate these back to $K$ we find $K_{\mathrm{crit}} = \mp\frac{1}{2}\log\left(\frac{3}{5}\right) \sim \pm 0.255\,412\ldots$ for the ferromagnetic and anti-ferromagnetic critical couplings, respectively. We can immediately see that since $|K_{\mathrm{crit}}| < |K_{\mathrm{min}}|$ it is now possible for the curve for a singly decorated $K(L)$ to cross the anti-ferromagnetic line for a suitable $\alpha$ value. In figure 5 we show $K(L)$ for $\alpha = 0.92$, where the three transition points may be seen. All of the transitions will display the KPZ [7, 8] exponents $\alpha = -1, \beta = \frac{1}{2}, \gamma = 2$. Thus, unlike the regular square lattice, the Ising model with singly decorated and competing bonds on random quadrangulations will display a sequence of one ferromagnetic and two anti-ferromagnetic transitions as the temperature is increased.

As we have noted, the decoration results are lattice dependent since they deal with critical temperatures. The ensemble of random quadrangulations we discussed above includes degenerate gluings of the squares along multiple edges, since the original $\phi^4$ graphs do not exclude self-energy and vertex correction diagrams. No analytical results are available for an ensemble of 'regular' random quadrangulations, where no multiple gluings are allowed, but the simulation of [4] gave an estimate of $K_{\mathrm{crit}} \sim 0.4$, which lies outside the range for multiple transitions with a single decoration. In this case we would *not* expect to see the sequence of transitions discussed above. However, doubly decorated bonds have a minimum of $K(L)$ at

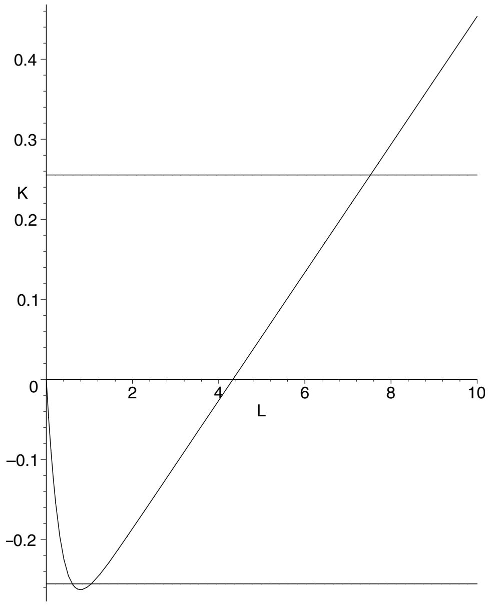


**Figure 5.** A plot of $K$ versus $L$ for a singly decorated bond, $n = 1$, $\alpha = 0.92$. The values of $K$ corresponding to the ferromagnetic $-\frac{1}{2}\log\frac{3}{5} \sim 0.255\,412\ldots$ and antiferromagnetic $-\frac{1}{2}\log\frac{5}{3} \sim -0.255\,412\ldots$ transitions on random quadrangulations are also shown. In this case a singly decorated bond is sufficient to induce a triple transition.

$K_{\min,2} \sim -0.5493\ldots$ and thus would give rise to multiple transitions for these regular random quadrangulations, just as they do for the regular square lattice Ising model.

It is also possible, of course, to elaborate the decoration procedure in various ways [9]. Introducing a higher spin $s$ which can take values $-S, -S+1, \ldots, S-1, S$ as the decorating spin and taking its interaction with the Ising spins to be

$$E = -\frac{L}{S} s\sigma \tag{10}$$

gives an effective coupling

$$K = -\alpha L + \frac{1}{2}\log\left[\frac{\sinh\left(\frac{(2S+1)L}{S}\right)}{(2S+1)\sinh\left(\frac{L}{S}\right)}\right] \tag{11}$$

(where we have again allowed for a direct anti-ferromagnetic bond). For this higher-spin decoration $K(L)$ still looks broadly similar to figures 3 and 5, but increasing $S$ has the effect of deepening the minimum in $K(L)$ and hence allowing multiple transitions where decoration by a spin-$\frac{1}{2}$ Ising spin would be insufficient.

The Ising models discussed here have been living on an annealed ensemble of random quadrangulations since the sum over graphs in equation (7) is at the level of the partition function and not the free energy. They therefore represent a very particular sort of annealed geometrical disorder. A natural elaboration of the discussion here is to consider the effects of similar, but quenched, geometrical disorder. Although there have been no investigations of the case of quenched ensembles of $\phi^4$ graphs or random quadrangulations, simulations of various spin models on quenched ensembles of $\phi^3$ graphs or their dual triangulations strongly suggest that there is little, if any, change in the critical couplings by comparison with the annealed ensemble [10]. The preceding discussion of the effects of the decoration–iteration transformation could therefore be carried over verbatim—on quenched random quadrangulations (allowing degenerate gluings) a competing anti-ferromagnetic and decorated ferromagnetic bonds would be expected to give rise to multiple transitions.

Other models with quenched geometrical disorder can also be subjected to the same treatment. A Penrose tiling is just such a case, since it is composed of rhombi and hence displays an anti-ferromagnetic transition. The critical couplings here are $K_{\mathrm{crit}} \sim \pm 0.418\,57\ldots$ [11] so, as for the square lattice Ising model and the regular random quadrangulations, higher decoration or higher spin decoration would be required to force the system to display multiple transitions. On the Penrose tiling all of the transitions would be in the Onsager universality class.

In summary, we have seen that geometrical disorder is no hindrance to carrying out a decoration–iteration transformation and inducing multiple transitions. The value of the critical coupling on (annealed) random quadrangulations means that a single decoration is sufficient to have an effect. It is likely that quenched random quadrangulations would show a similar effect. It might be of some interest to explore the effect of some of the other 'classic' transformations, such as decoration–iteration in field or star–triangle, for the Ising model coupled to 2D quantum gravity (i.e. living on planar random graphs or their duals) since, at least implicitly, the solution in field is available.

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